A conservative corrected SPH for wave propagation

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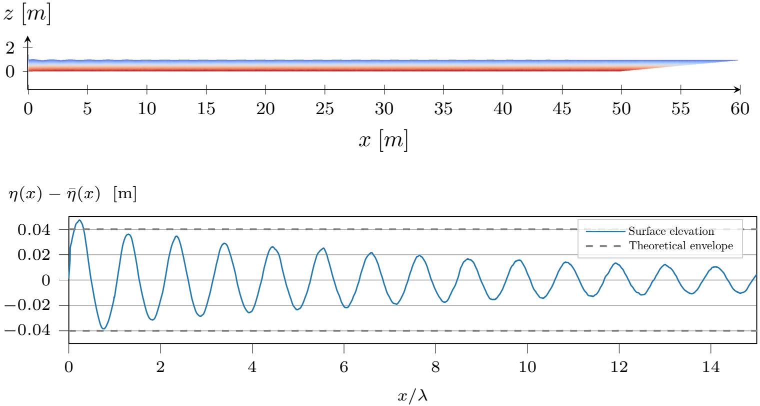
- Waves simulated with SPH can exhibit an excessive non-physical decay
- The decay is linked to the wavelength λ, thus shorter waves propagate less



SPH parameters:

- $\Delta p = 1/128m$
- Lennard-Jones boundary, periodic in Y direction.

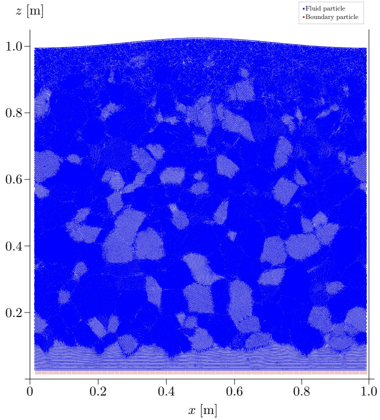
The domain is $8h$ thick in the y direction.

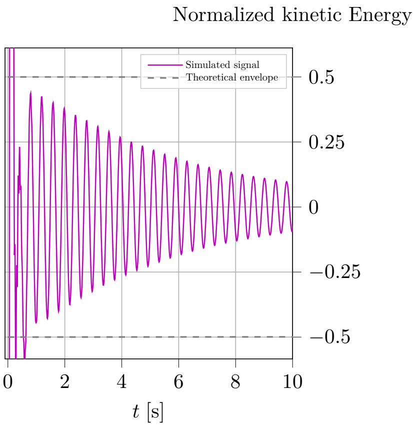
The excessive decay can also be observed in the kinetic energy of a standing wave [1].

SPH parameters:

- $\Delta p = 1/256m$
- Periodic conditions (side walls) Dynamic boundary (bottom)

The domain is $12h$ thick in the y direction.





[1] Domain settings as in Antuono et al. 2011, but in 3D here.

We fit the envelope of the kinetic energy in the simulated standing wave with an exponential function

$$E_{sim}(t) = E(t_0)e^{-\beta t}, \qquad (1)$$

where β is the decay coefficient. Then we compare it with the theoretical (slow) decay

$$E_{real}(t) = E(t_0)e^{-\beta_0 t} \qquad (2)$$

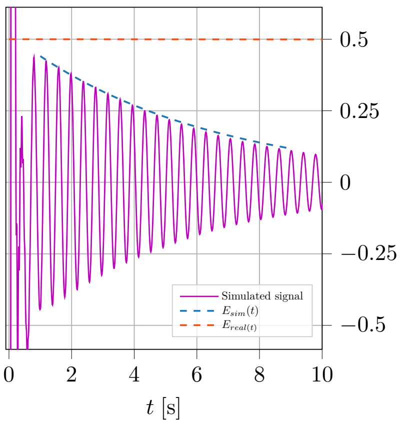
which is due to the viscosity of the fluid, and has a decay coefficient $\beta_0 = 4\nu k^2$.
We express the conservation of energy using the ratio β/β_0.

In this case: $\frac{\beta}{\beta_0} = \frac{0.164}{1.58\ 10^{-4}} = 1.04\ 10^3$

In terms of energy drop:

$$E_{sim}(4.2) = 0.5\ E(0) = E_{real}(4387.1)$$

Normalized kinetic Energy



The SPH scheme that we use as reference is based on the Navier-Stokes equations, for mass consevation:

$$\frac{D\rho_i}{Dt} = \sum_j \mathbf{u}_{ij} \cdot \mathbf{x}_{ij} F_{ij} m_j + \xi h c_0 \sum_j \Psi_{ij} F_{ij} m_j. \quad (3)$$

where we have Molteni and Colagrossi density diffusion, and momentum conservation:

$$\frac{D\mathbf{u}_i}{Dt} = \sum_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \Pi_{ij} \right) \mathbf{x}_{ij} F_{ij} m_j + \mathbf{g}. \quad (4)$$

We use a Wendland kernel and a smoothing factor $\alpha_s = h/\Delta p = 1.3$. In the following we will refer to this SPH formulation as Standard SPH, SSPH.

All simulations are run in GPUSPH in single numerical precision.

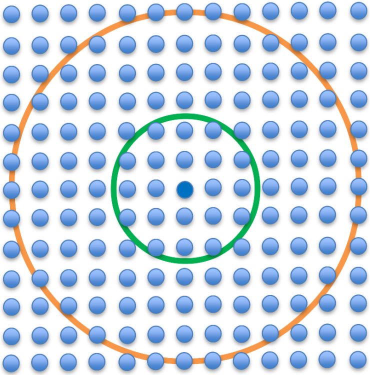
The main current approaches are:

- **Larger kernel support**: increasing the numbers of neighbors improves conservation of energy.[2] Larger numbers are needed at high Reynolds numbers.

  $$\text{e.g } \alpha_s \approx 3 \text{ would be needed for } Re \propto 10^3$$

  Simulation time is strongly affected.

- **Kernel Gradient Corrections**: can reduce the decay[3] while keeping a limited number of neighbors, but can affect stability and conservation of momentum.



[3] Colagrossi et al. (2013), Smoothed-particle-hydrodynamics modeling of dissipation mechanisms in gravity waves, Physical review E.

[3] Wen et al. (2019), An improved SPH model for turbulent hydrodynamics of a 2D oscillating water chamber, Ocean Engineering

A KGC improves the particle approximation of a gradient field. It is obtained by multiplying the kernel gradient by a corrective coefficient A^{-1}

$$\tilde{\nabla} W_{ij} = \mathbf{A}_i^{-1} \nabla W_{ij} \tag{5}$$

Where A is a matrix (3x3 in a 3D space) defined as:

$$\mathbf{A}_i = \sum_j \nabla W_{ij} \otimes (\mathbf{x}_j - \mathbf{x}_i) V_j \tag{6}$$

This matrix is defined for each particle and depends on the position and volumes of the neighbors.

A matrix inversion is required for each particle.

- **Conservation of momentum**: correction coefficients depend on local neighborhood properties and differ from particle to particle.

$$\frac{D\mathbf{u}_i}{Dt} = \sum_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \Pi_{ij} \right) \mathbf{A}_i^{-1} \mathbf{x}_{ij} F_{ij} m_j + \mathbf{g} \tag{7}$$

  Interactions are therefore not symmetric and momentum conservation is not guaranteed.

- **Stability and quality of simulations**: irregular particle distributions or incomplete supports may affect the conditioning of the $\mathbf{A}$ matrix coefficients, impacting the stability and the quality of the simulation, as the energy conservation itself.

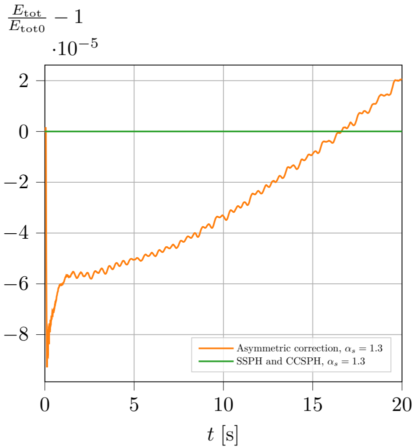
Symmetric interactions can be guaranteed by using a correction coefficient defined by averaging the coefficients associated with the two interacting particles

$$\mathbf{B}_{ij} = \frac{1}{2}(\mathbf{A}_i + \mathbf{A}_j) \qquad (8)$$

with the corrected equation of momentum becoming

$$\frac{D\mathbf{u}_i}{Dt} = \sum_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} + \Pi_{ij}\right) \mathbf{B}_{ij}^{-1}\mathbf{x}_{ij} F_{ij} m_j + \mathbf{g} \qquad (9)$$

The total energy (Kinetic + Potential) shows that conservation is restored.



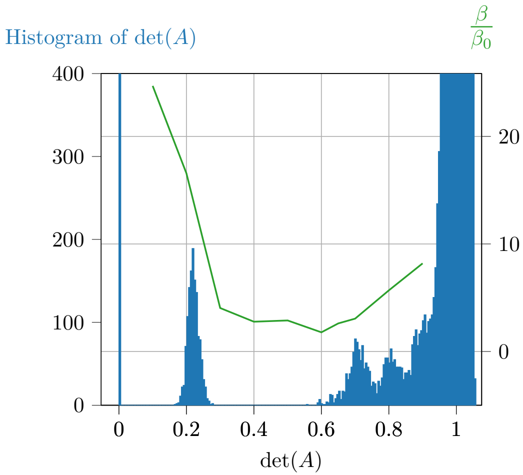
Irregular particle distributions can be found at the boundaries of the domain.

We assign $\mathbf{A} = \mathbf{I}$ to boundary particles and their neighbors.

In other parts of the domain e.g. at and close to the free surface, we cannot deliberately suppress the correction.

From a parameter study based on the β/β_0 ratio we chose the value of a threshold $det(\mathbf{A})_{th}$, below which we assign $\mathbf{A} = \mathbf{I}$.

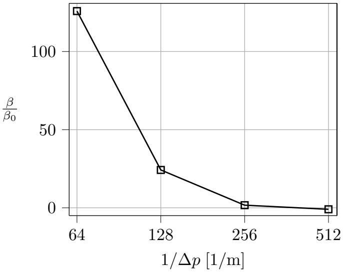
The optimal value is $det(\mathbf{A})_{th} = 0.6$, that is also reflected on the histogram of $det(\mathbf{A})$.



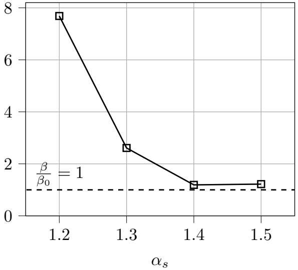
We will refer to this conservative corrected formulation as CCSPH.

Convergence test were performed with respect to Δp and α_s, looking at the ratio β/β_0, and convergence is obtained in both cases.

Good results are obtained for for $\Delta p = 1/256$ or $\alpha_s = 1.4$, where the error between numerical and analytical solutions is comparable to machine precision. Some irregularities in the convergence trend appear due to numerical reasons at higher values.

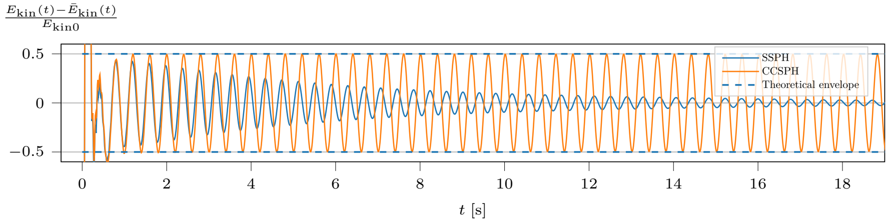


$\alpha_s = 1.3$

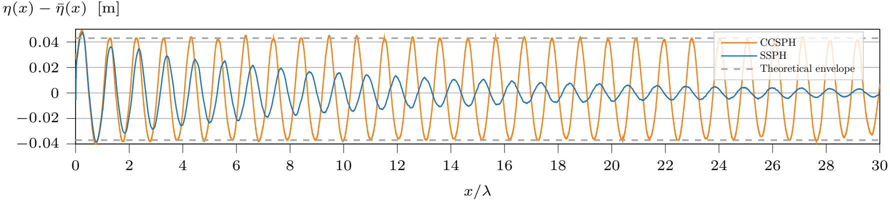


$\Delta p = 1/256$ [m]

Here we compare the results obtained for SSPH and CCSPH, both with a smoothing ratio $\alpha_s = 1.3$.
For the standing wave the kinetic energy:



And the surface elevation of the progressive waves:



KGC can eliminate the excessive decay similarly to increasing α_s, but keeping lower computational load. We use the standing wave problem run on an NVIDIA Titan XP GPU, and simulate a time of 0.1s:

1) SSPH and $\alpha_s = 1.3$ delivers $\beta/\beta_0 = 1043.3$ and takes 153s
2) CCSPH and $\alpha_s = 1.3$ delivers $\beta/\beta_0 = 2.61$ and takes 344s (2.25 times case 1)
3) CCSPH and $\alpha_s = 1.4$ delivers $\beta/\beta_0 = 1.19$ and takes 369s (2.41 times case 1)
4) SSPH and $\alpha_s = 3.0$ delivers $\beta/\beta_0 = 17.25$ and takes 1248s (8.11 times case 1)

- We developed a corrected SPH model (CCSPH), implemented on GPUSPH, that does not experience the excessive decay of simulated water waves while maintaining low computational load.
- Convergence was assessed by simulating a standing wave with various particle resolutions and smoothing factors.
- We have shown the simulation of a 30 wavelengths wave train with no excessive decay. The limit of 30 wavelengths was only chosen in relation to the time required by the simulations, while the results that we have obtained do not suggest any limit on the propagation distance.